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LARGE ELASTIC DEFORMATIONS OF ISOTROPIC MATERIALS

VI. FURTHER RESULTS IN THE THEORY OF TORSION, SHEAR AND FLEXURE

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The forces necessary to produce certain simple types of deformation in a tube of incompressible, highly elastic material, isotropic in its undeformed state, are discussed. The first type of deformation may be considered to be produced by the following three successive simpler deformations:

(i) a uniform simple extension,

(ii) a uniform inflation of the tube, in which its length remains constant, and

(iii) a uniform simple torsion, in which planes perpendicular to the axis of the tube are rotated in their own plane through an angle proportional to their distance from one end of the tube.

Certain special cases of this deformation are considered in greater detail employing a simple stored-energy function of the form

$$W = C_1(I_1-3)+C_2(I_2-3),$$

where C_1 and C_2 are physical constants for the material and I_1 and I_2 are the strain invariants.

The second type of deformation considered is that in which the simpler deformations (i) and (ii) mentioned above are followed successively by simple shears about the axis of the tube and parallel to it. The forces which must be applied are calculated for the simple form of stored-energy function given above.

Finally, the simultaneous simple flexure and uniform extension normal to the plane of flexure of a thick sheet is discussed, and a number of the results obtained in a previous paper (Rivlin 1949*b*) are generalized.

1. Introduction

In part I (Rivlin 1948*a*) of this series of papers, the notion of an incompressible, neo-Hookean material was introduced and the stress-strain relations, equations of motion and boundary conditions were given for a body of such material. In part III (Rivlin 1948*b*) the surface tractions necessary to produce various simple types of deformation in a right-circular cylinder and in a tube of circular cross-section of incompressible, neo-Hookean material were calculated. In part IV (Rivlin 1948*c*), the stress-strain relations, equations of motion and boundary conditions for *any* incompressible material, which is isotropic in its undeformed state and for which the stored-energy function W is a function of the two strain invariants I_1 and I_2, were derived in a form suitable for the calculation of the forces necessary to produce a specified deformation of a body of the material, without any explicit form for W being assumed. These equations were applied to the calculation of the forces necessary to produce simple torsion in a cylinder of the material (Rivlin 1948*c*, 1949*a*). It was found that, apart from an arbitrary hydrostatic pressure applied over the whole surface of the cylinder, azimuthal and normal surface tractions must be applied to the ends of the cylinder. The distribution of these and their variation with amount of torsion depend on the precise form of the stored-energy function. It is of interest that Poynting (1909, 1913)* carried out

* I am indebted to Dr H. A. Daynes for drawing my attention to Poynting's work.

experiments in which he measured the lengthening of a steel wire and of a rubber rod on twisting. His theoretical discussions of the effect are, however, inadequate.

In the present paper, the theory is applied to the calculation of the forces necessary to produce certain simple types of deformation in a uniform tube of circular cross-section. First, the simultaneous extension, inflation and torsion of the tube is considered, without making any specific assumptions regarding the form of the stored-energy function, other than those implicit in the incompressibility of the material and its isotropy in the undeformed state. The distribution of applied surface forces necessary to produce such a deformation is considered and the resultant torsional couple, longitudinal force and inflating pressure are obtained. This result is specialized to the case of a cylindrical rod in § 6 and, in § 7, to the problem of a tube turned inside out, so that it is maintained in a state of deformation without any external forces being applied. In § 8, the simultaneous extension and torsion of a tube, without any inflating pressure being applied, is discussed. For the discussions of §§ 7 and 8, the simple form of stored-energy function used by Mooney (1940) and given by

$$W = C_1(I_1-3)+C_2(I_2-3) \tag{1·1}$$

is employed. Here C_1 and C_2 are physical constants characterizing the material and I_1 and I_2 are strain invariants defined by

$$I_1 = \lambda_1^2+\lambda_2^2+\lambda_3^2 \quad \text{and} \quad I_2 = \frac{1}{\lambda_1^2}+\frac{1}{\lambda_2^2}+\frac{1}{\lambda_3^2}, \tag{1·2}$$

where λ_1, λ_2 and λ_3 are the principal extension ratios for the deformation.

From the point of view of the theory of small, but finite, deformations, the form (1·1) for the stored-energy function has the special significance that it represents the most general form which can be taken by the stored-energy function for an incompressible material, which is isotropic in its undeformed state, if terms of higher degree than the third in the principal extensions $(\lambda_1-1, \lambda_2-1, \lambda_3-1)$ are neglected.

In §§ 9 to 13, it is considered that the tube is first stretched over a rigid cylinder and held between this and a rigid outer coaxial cylindrical shell. It is then subjected simultaneously to shearing deformations resulting from the movement of one rigid cylinder with respect to the other along their common axis and the rotation of one cylinder with respect to the other about their common axis. The surface forces which must be applied in order to produce this deformation are calculated. In order to carry this calculation through, the case when W has the simple form (1·1) is considered, since, unless the form of W as a function of I_1 and I_2 is explicitly given, the displacements undergone by each point of the body cannot be calculated from the displacements of the rigid cylinders. The edge effects resulting from the finite length of the tube are not discussed here, but it is hoped to do so in a later paper.

In §§ 14 to 16, the problem of the simultaneous simple flexure and uniform extension normal to the plane of flexure of a cuboid is considered. This forms an extension to the discussion of the simple flexure of a cuboid in an earlier paper (Rivlin 1949 *b*). There the discussion was specialized at an early stage to the case when W has the form given in (1·1). In the present paper the discussion is carried further without specializing the form of W.

2. The basic formulae

We consider the deformation of an incompressible material, which is isotropic in its undeformed state, in which a point, which is initially at (x', y', z') in the rectangular Cartesian co-ordinate system (x', y', z'), moves to (ξ', η', ζ'). If W is the stored-energy function for the material, considered as a function of the strain invariants I_1 and I_2, then the stress components $t_{x'x'}, t_{y'y'}, \ldots, t_{x'y'}$ in the co-ordinate system (x', y', z') are given (Rivlin 1948*c*, equations (4·6)) by

$$\left.\begin{aligned} t_{x'x'} &= 2\left[(1+2\epsilon'_{x'x'})\frac{\partial W}{\partial I_1} - \{(1+2\epsilon'_{y'y'})(1+2\epsilon'_{z'z'}) - \epsilon'^2_{y'z'}\}\frac{\partial W}{\partial I_2} + I_2\frac{\partial W}{\partial I_2}\right] + p, \quad \text{etc.,} \\ \text{and} \quad t_{y'z'} &= 2\left[\epsilon'_{y'z'}\frac{\partial W}{\partial I_1} - \{\epsilon'_{x'y'}\epsilon'_{z'x'} - (1+2\epsilon'_{x'x'})\epsilon'_{y'z'}\}\frac{\partial W}{\partial I_2}\right], \quad \text{etc.,} \end{aligned}\right\} \qquad (2·1)$$

$$\left.\begin{aligned} \text{where} \quad 1+2\epsilon'_{x'x'} &= \xi'^2_{x'} + \xi'^2_{y'} + \xi'^2_{z'}, \quad \text{etc.,} \\ \text{and} \quad \epsilon'_{y'z'} &= \eta'_{x'}\zeta'_{x'} + \eta'_{y'}\zeta'_{y'} + \eta'_{z'}\zeta'_{z'}, \quad \text{etc.} \end{aligned}\right\} \qquad (2·2)$$

Equations (2·1) can be shown to be the equivalents, in Lagrangian form and for an incompressible material, of those given in Eulerian form by Signorini (1943).

We shall transform these equations into a cylindrical polar co-ordinate system (r, θ, z). Let us assume that the cylindrical polar co-ordinates of the point (x', y', z') are (r, θ, z), and those of the point (ξ', η', ζ'), to which it moves in the deformation, are (ρ, ϑ, ζ). We shall choose the co-ordinate system (x', y', z') so that its origin is at the point $(\rho_0, \vartheta_0, \zeta_0)$ and its axes are in the radial, azimuthal and longitudinal directions respectively at that point. Then

$$\left.\begin{aligned} \xi' &= \rho\cos(\vartheta - \vartheta_0) - \rho_0, \\ \eta' &= \rho\sin(\vartheta - \vartheta_0) \\ \text{and} \quad \zeta' &= \zeta - \zeta_0. \end{aligned}\right\} \qquad (2·3)$$

$$\left.\begin{aligned} \text{Also,} \quad x' &= r\cos(\theta - \vartheta_0) - \rho_0, \\ y' &= r\sin(\theta - \vartheta_0) \\ \text{and} \quad z' &= z - \zeta_0. \end{aligned}\right\} \qquad (2·4)$$

From (2·4), writing $\vartheta' = \vartheta - \vartheta_0$ and $\phi = \vartheta - \theta$, we have

$$\left.\begin{aligned} \frac{\partial}{\partial x'} &= \cos(\vartheta' - \phi)\frac{\partial}{\partial r} - \frac{1}{r}\sin(\vartheta' - \phi)\frac{\partial}{\partial \theta}, \\ \frac{\partial}{\partial y'} &= \sin(\vartheta' - \phi)\frac{\partial}{\partial r} + \frac{1}{r}\cos(\vartheta' - \phi)\frac{\partial}{\partial \theta} \\ \text{and} \quad \frac{\partial}{\partial z'} &= \frac{\partial}{\partial z}. \end{aligned}\right\} \qquad (2·5)$$

If the point considered is at $(\rho_0, \vartheta_0, \zeta_0)$ in the deformed state, so that $\vartheta' = 0$, equations (2·5) become

$$\left.\begin{aligned} \frac{\partial}{\partial x'} &= \cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\sin\phi\frac{\partial}{\partial \theta}, \\ \frac{\partial}{\partial y'} &= -\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\phi\frac{\partial}{\partial \theta} \\ \text{and} \quad \frac{\partial}{\partial z'} &= \frac{\partial}{\partial z}. \end{aligned}\right\} \qquad (2·6)$$

From (2·3) and (2·6), we have, for this point,

$$\left.\begin{aligned} \xi'_{x'} &= a_{11}l+a_{12}m, & \eta'_{x'} &= a_{21}l+a_{22}m, & \zeta'_{x'} &= a_{31}l+a_{32}m,\\ \xi'_{y'} &= a_{12}l-a_{11}m, & \eta'_{y'} &= a_{22}l-a_{21}m, & \zeta'_{y'} &= a_{32}l-a_{31}m,\\ \xi'_{z'} &= a_{13}, & \eta'_{z'} &= a_{23}, & \zeta'_{z'} &= a_{33}, \end{aligned}\right\} \tag{2·7}$$

where $$l = \cos\phi \quad \text{and} \quad m = \sin\phi, \tag{2·8}$$

and a_{ij} ($i = 1, 2, 3$ and $j = 1, 2, 3$) is given by

$$\|a_{ij}\| = \begin{pmatrix} \rho_r & \rho_\theta/r & \rho_z \\ \rho\vartheta_r & \rho\vartheta_\theta/r & \rho\vartheta_z \\ \zeta_r & \zeta_\theta/r & \zeta_z \end{pmatrix}. \tag{2·9}$$

From (2·7) and (2·2), we have, employing the double-suffix notation for summation over $i = 1, 2, 3$, and denoting $\epsilon'_{x'x'}, \epsilon'_{y'y'}, \ldots, \epsilon'_{x'y'}$ by $\epsilon'_{rr}, \epsilon'_{\theta\theta}, \ldots, \epsilon'_{r\theta}$, we have

$$\left.\begin{aligned} 1+2\epsilon'_{rr} &= a_{1i}a_{1i}, \quad 1+2\epsilon'_{\theta\theta} = a_{2i}a_{2i}, \quad 1+2\epsilon'_{zz} = a_{3i}a_{3i},\\ \epsilon'_{\theta z} &= a_{2i}a_{3i}, \quad \epsilon'_{zr} = a_{3i}a_{1i} \quad \text{and} \quad \epsilon'_{r\theta} = a_{1i}a_{2i}. \end{aligned}\right\} \tag{2·10}$$

From (2·10), it can be shown algebraically that

$$\left.\begin{aligned} (1+2\epsilon'_{\theta\theta})\,(1+2\epsilon'_{zz})-\epsilon'^2_{\theta z} &= A_{1i}A_{1i}, \quad \text{etc.,}\\ \text{and} \qquad \epsilon'_{zr}\epsilon'_{r\theta}-(1+2\epsilon'_{rr})\,\epsilon'_{\theta z} &= A_{2i}A_{3i}, \quad \text{etc.} \end{aligned}\right\} \tag{2·11}$$

The terms on the right-hand sides of equations (2·11) are similar to those on the right-hand sides of equations (2·10), a_{ij} being replaced by A_{ij}, where A_{ij} ($i = 1, 2, 3$; $j = 1, 2, 3$) is the determinant of the minor of a_{ij} in the matrix (2·9).

Denoting the stress components $t_{x'x'}, t_{y'y'}, \ldots, t_{x'y'}$ by $t_{\rho\rho}$, $t_{\vartheta\vartheta}$, $t_{\zeta\zeta}$, $t_{\vartheta\zeta}$, $t_{\zeta\rho}$ and $t_{\rho\vartheta}$ respectively, we obtain, from (2·1),

$$\left.\begin{aligned} t_{\rho\rho} &= 2\left[(1+2\epsilon'_{rr})\frac{\partial W}{\partial I_1}-\{(1+2\epsilon'_{\theta\theta})\,(1+2\epsilon'_{zz})-\epsilon'^2_{\theta z}\}\frac{\partial W}{\partial I_2}+I_2\frac{\partial W}{\partial I_2}\right]+p, \quad \text{etc.,}\\ \text{and} \qquad t_{\vartheta\zeta} &= 2\left[\epsilon'_{\theta z}\frac{\partial W}{\partial I_1}-\{\epsilon'_{zr}\epsilon'_{r\theta}-(1+2\epsilon'_{rr})\,\epsilon'_{\theta z}\}\frac{\partial W}{\partial I_2}\right], \quad \text{etc.} \end{aligned}\right\} \tag{2·12}$$

From (2·10) and (2·11) respectively, we can readily obtain, in cylindrical polar coordinates, the expressions for the strain invariants I_1 and I_2, given in the previous paper (Rivlin 1948*c*, equations (3·4)). We have

$$I_1 = a_{1i}a_{1i}+a_{2i}a_{2i}+a_{3i}a_{3i} \quad \text{and} \quad I_2 = A_{1i}A_{1i}+A_{2i}A_{2i}+A_{3i}A_{3i}. \tag{2·13}$$

The incompressibility condition $\partial(\xi', \eta', \zeta')/\partial(x', y', z') = 1$ becomes, with (2·7),

$$\det a_{ij} = 1. \tag{2·14}$$

Let $R_{\nu'}$, $\Theta_{\nu'}$, $Z_{\nu'}$ be the components of the surface traction acting at the point (ρ, ϑ, ζ) in the deformed state, which are in the radial, azimuthal and longitudinal directions respectively at (ρ, ϑ, ζ). These are measured per unit area of the surface to which they are applied in its deformed state. They are given in terms of the stress components $t_{\rho\rho}$, $t_{\vartheta\vartheta}$, …, $t_{\rho\vartheta}$ by

$$\left.\begin{aligned} R_{\nu'} &= t_{\rho\rho}\cos(\rho, \nu')+t_{\rho\vartheta}\cos(\vartheta, \nu')+t_{\zeta\rho}\cos(\zeta, \nu'),\\ \Theta_{\nu'} &= t_{\rho\vartheta}\cos(\rho, \nu')+t_{\vartheta\vartheta}\cos(\vartheta, \nu')+t_{\vartheta\zeta}\cos(\zeta, \nu')\\ \text{and} \qquad Z_{\nu'} &= t_{\zeta\rho}\cos(\rho, \nu')+t_{\vartheta\zeta}\cos(\vartheta, \nu')+t_{\zeta\zeta}\cos(\zeta, \nu'), \end{aligned}\right\} \tag{2·15}$$

where (ρ, ν'), (ϑ, ν') and (ζ, ν') are the angles of inclination of the normal ν' to the deformed surface at the point considered to the radial, azimuthal and longitudinal directions respectively at that point.

In order to calculate the surface traction from equation (2·15), it is necessary to know the position of the surface in its deformed state.

In a previous paper (Rivlin 1948*c*, equations (11·2)) it has been shown that the components of the surface traction X'_ν, Y'_ν, Z'_ν in a rectangular Cartesian co-ordinate system (x', y', z') are given by

$$X'_\nu = \frac{\partial W}{\partial I_1}\left[\frac{\partial I_1}{\partial \xi'_{x'}}\cos(x',\nu) + \frac{\partial I_1}{\partial \xi'_{y'}}\cos(y',\nu) + \frac{\partial I_1}{\partial \xi'_{z'}}\cos(z',\nu)\right] + \frac{\partial W}{\partial I_2}\left[\frac{\partial I_2}{\partial \xi'_{x'}}\cos(x',\nu) + \frac{\partial I_2}{\partial \xi'_{y'}}\cos(y',\nu) + \frac{\partial I_2}{\partial \xi'_{z'}}\cos(z',\nu)\right] + p\left[\frac{\partial \tau}{\partial \xi'_{x'}}\cos(x',\nu) + \frac{\partial \tau}{\partial \xi'_{y'}}\cos(y',\nu) + \frac{\partial \tau}{\partial \xi'_{z'}}\cos(z',\nu)\right], \text{ etc.} \tag{2·16}$$

where (x', ν), (y', ν) and (z', ν) denote the inclinations of the normal ν to the surface at the point considered, in its undeformed state, to the axes x', y' and z' respectively. X'_ν, Y'_ν and Z'_ν are measured per unit area of the surface in its undeformed state. τ is defined by

$$\tau = \frac{\partial(\xi', \eta', \zeta')}{\partial(x', y', z')}. \tag{2·17}$$

$\partial I_1/\partial \xi'_{x'}$, $\partial I_1/\partial \xi'_{y'}$, ... and $\partial I_2/\partial \xi'_{x'}$, $\partial I_2/\partial \xi'_{y'}$, ...

are given by

$$\frac{\partial I_1}{\partial \xi'_{x'}} = 2\xi'_{x'}, \quad \frac{\partial I_1}{\partial \xi'_{y'}} = 2\xi'_{y'}, \text{ etc.,} \tag{2·18}$$

and

$$\left.\begin{aligned} \frac{\partial I_2}{\partial \xi'_{x'}} &= 2\left[\eta'_{y'}\frac{\partial \tau}{\partial \zeta'_{z'}} + \zeta'_{z'}\frac{\partial \tau}{\partial \eta'_{y'}} - \eta'_{z'}\frac{\partial \tau}{\partial \zeta'_{y'}} - \zeta'_{y'}\frac{\partial \tau}{\partial \eta'_{z'}}\right], \\ \frac{\partial I_2}{\partial \xi'_{y'}} &= 2\left[\eta'_{z'}\frac{\partial \tau}{\partial \zeta'_{x'}} + \zeta'_{x'}\frac{\partial \tau}{\partial \eta'_{z'}} - \eta'_{x'}\frac{\partial \tau}{\partial \zeta'_{z'}} - \zeta'_{z'}\frac{\partial \tau}{\partial \eta'_{x'}}\right], \text{ etc.} \end{aligned}\right\} \tag{2·19}$$

Now, as before, we can choose the axes (x', y', z') to coincide with the radial, azimuthal and longitudinal directions respectively at the point (ρ, ϑ, ζ). Then, we can write

$$(x', \nu) = (\rho, \nu), \quad (y', \nu) = (\vartheta, \nu) \quad \text{and} \quad (z', \nu) = (\zeta, \nu), \tag{2·20}$$

$$X'_\nu = R_\nu, \quad Y'_\nu = \Theta_\nu \quad \text{and} \quad Z'_\nu = Z_\nu. \tag{2·21}$$

R_ν, Θ_ν and Z_ν are the components of the surface traction corresponding to $R_{\nu'}$, $\Theta_{\nu'}$ and $Z_{\nu'}$ respectively, but are measured per unit area of the surface to which they are applied in its undeformed state.

From (2·18) and (2·7),

$$\partial I_1/\partial \xi'_{x'} = 2(a_{11}l + a_{12}m), \quad \partial I_1/\partial \xi'_{y'} = 2(a_{12}l - a_{11}m), \text{ etc.} \tag{2·22}$$

From (2·7) we have

$$\left.\begin{aligned} \partial\tau/\partial\xi'_{x'} &= A_{11}l + A_{12}m, & \partial\tau/\partial\xi'_{y'} &= A_{12}l - A_{11}m, & \partial\tau/\partial\xi'_{z'} &= A_{13}, \\ \partial\tau/\partial\eta'_{x'} &= A_{21}l + A_{22}m, & \partial\tau/\partial\eta'_{y'} &= A_{22}l - A_{21}m, & \partial\tau/\partial\eta'_{z'} &= A_{23}, \\ \partial\tau/\partial\zeta'_{x'} &= A_{31}l + A_{32}m, & \partial\tau/\partial\zeta'_{y'} &= A_{32}l - A_{31}m, & \partial\tau/\partial\zeta'_{z'} &= A_{33}. \end{aligned}\right\} \tag{2·23}$$

From (2·23), (2·7) and (2·19) we obtain

$$\left.\begin{aligned}
\tfrac{1}{2}\partial I_2/\partial \xi'_{x'} &= l(a_{22}A_{33}+a_{33}A_{22}-a_{23}A_{32}-a_{32}A_{23})+m(a_{23}A_{31}+a_{31}A_{23}-a_{21}A_{33}-a_{33}A_{21}),\\
\tfrac{1}{2}\partial I_2/\partial \xi'_{y'} &= l(a_{23}A_{31}+a_{31}A_{23}-a_{21}A_{33}-a_{33}A_{21})-m(a_{22}A_{33}+a_{33}A_{22}-a_{23}A_{32}-a_{32}A_{23}),\\
\tfrac{1}{2}\partial I_2/\partial \xi'_{z'} &= a_{21}A_{32}+a_{32}A_{21}-a_{22}A_{31}-a_{31}A_{22},\\
\tfrac{1}{2}\partial I_2/\partial \eta'_{x'} &= l(a_{32}A_{13}+a_{13}A_{32}-a_{33}A_{12}-a_{12}A_{33})+m(a_{33}A_{11}+a_{11}A_{33}-a_{31}A_{13}-a_{13}A_{31}),\\
\tfrac{1}{2}\partial I_2/\partial \eta'_{y'} &= l(a_{33}A_{11}+a_{11}A_{33}-a_{31}A_{13}-a_{13}A_{31})-m(a_{32}A_{13}+a_{13}A_{32}-a_{33}A_{12}-a_{12}A_{33}),\\
\tfrac{1}{2}\partial I_2/\partial \eta'_{z'} &= a_{31}A_{12}+a_{12}A_{31}-a_{32}A_{11}-a_{11}A_{32},\\
\tfrac{1}{2}\partial I_2/\partial \zeta'_{x'} &= l(a_{12}A_{23}+a_{23}A_{12}-a_{13}A_{22}-a_{22}A_{13})+m(a_{13}A_{21}+a_{21}A_{13}-a_{11}A_{23}-a_{23}A_{11}),\\
\tfrac{1}{2}\partial I_2/\partial \zeta'_{y'} &= l(a_{13}A_{21}+a_{21}A_{13}-a_{11}A_{23}-a_{23}A_{11})-m(a_{12}A_{23}+a_{23}A_{12}-a_{13}A_{22}-a_{22}A_{13}),\\
\tfrac{1}{2}\partial I_2/\partial \zeta'_{z'} &= a_{11}A_{22}+a_{22}A_{11}-a_{12}A_{21}-a_{21}A_{12}.
\end{aligned}\right\} \quad (2\cdot24)$$

For the equilibrium of an infinitesimal element of volume situated at (ρ, ϑ, ζ) in the deformed state of the body, we have, applying Newton's second law in the radial, azimuthal and longitudinal directions at that point,

$$\left.\begin{aligned}
\frac{\partial t_{\rho\rho}}{\partial \rho}+\frac{1}{\rho}\frac{\partial t_{\rho\vartheta}}{\partial \vartheta}+\frac{\partial t_{\zeta\rho}}{\partial \zeta}+\frac{t_{\rho\rho}-t_{\vartheta\vartheta}}{\rho}+R' &= 0,\\
\frac{\partial t_{\rho\vartheta}}{\partial \rho}+\frac{1}{\rho}\frac{\partial t_{\vartheta\vartheta}}{\partial \vartheta}+\frac{\partial t_{\zeta\vartheta}}{\partial \zeta}+\frac{2}{\rho}t_{\rho\vartheta}+\Theta' &= 0\\
\text{and}\qquad \frac{\partial t_{\zeta\rho}}{\partial \rho}+\frac{1}{\rho}\frac{\partial t_{\vartheta\zeta}}{\partial \vartheta}+\frac{\partial t_{\zeta\zeta}}{\partial \zeta}+\frac{1}{\rho}t_{\zeta\rho}+Z' &= 0,
\end{aligned}\right\} \quad (2\cdot25)$$

where R', Θ' and Z' are the components, in the radial, azimuthal and longitudinal directions respectively at the point (ρ, ϑ, ζ), of the body force acting at the point per unit volume of the deformed body.

SIMULTANEOUS EXTENSION, INFLATION AND TORSION OF A CYLINDRICAL TUBE

3. Definition of the deformation

In this part of the paper, we shall consider the simultaneous extension, inflation and torsion of a cylindrical tube of incompressible, highly elastic material, isotropic in its undeformed state, which has length l and external and internal radii a_1 and a_2 respectively in the undeformed state. In order to be precise, we shall assume that the final deformation is produced by the following three successive deformations:

(i) a uniform simple extension of extension ratio λ;

(ii) a uniform inflation of the tube in which its length remains constant and its external and internal radii change to $\mu_1 a_1$ and $\mu_2 a_2$ respectively;

(iii) a uniform simple torsion in which planes perpendicular to the axis of the tube are rotated in their own plane through an angle proportional to the distance of the plane considered from one end, the constant of proportionality being ψ.

Let us take as reference frame a cylindrical polar co-ordinate system (r, θ, z), the origin of which is at the centre of one end of the tube and in which the z-axis is along the axis of the tube. Then, in the simple extension and inflation, the point initially at (r, θ, z) moves to $(\rho, \theta, \lambda z)$, where ρ is a function of r only, which will be determined later. Again, in the torsion, the point moves to $(\rho, \theta+\psi\lambda z, \lambda z)$. The form of ρ, as a function of r, can be determined from the incompressibility condition (2·14) as

$$\lambda\rho\rho_r/r = 1, \tag{3·1}$$

i.e.

$$\rho = \left[\frac{1}{\lambda}(r^2+K)\right]^{\frac{1}{2}}, \tag{3·2}$$

where K is a constant of integration.

For compactness, we shall employ the notation

$$\rho = \mu r \quad \text{and} \quad \rho_r = 1/\lambda\mu. \tag{3·3}$$

Since $\rho = \mu_1 a_1$ when $r = a_1$, and $\mu_2 a_2$ when $r = a_2$, we obtain, from (3·2),

$$K = a_1^2(\lambda\mu_1^2-1) = a_2^2(\lambda\mu_2^2-1). \tag{3·4}$$

If the point initially at (r, θ, z) moves to (ρ, ϑ, ζ) in the deformation, then ρ is given by (3·2) and

$$\vartheta = \theta+\psi\lambda z \quad \text{and} \quad \zeta = \lambda z. \tag{3·5}$$

From (3·4), it is seen that if λ is positive, K may be either positive or negative, accordingly as $\lambda\mu_1^2$ is greater than or less than unity. If λ is negative, then K is negative and $\mu_2^2 a_2^2 > \mu_1^2 a_1^2$. This means that the tube is turned inside out.

If λ is positive, then (r^2+K) is positive for all values of r between a_1 and a_2, and if λ is negative, (r^2+K) is negative over this range of r.

4. The stress-strain relations and equations of equilibrium

For the deformation considered, we have, from (3·3), (3·5) and (2·9),

$$\left.\begin{aligned} a_{11} &= 1/\lambda\mu, & a_{12} &= 0, & a_{13} &= 0, \\ a_{21} &= 0, & a_{22} &= \mu, & a_{23} &= \psi\lambda\mu r, \\ a_{31} &= 0, & a_{32} &= 0, & a_{33} &= \lambda. \end{aligned}\right\} \tag{4·1}$$

From (4·1) and (2·10) the components of strain $\epsilon'_{rr}, \epsilon'_{\theta\theta}, \ldots, \epsilon'_{r\theta}$, in the cylindrical polar co-ordinate system (r, θ, z), are given by

$$\left.\begin{aligned} 1+2\epsilon'_{rr} = 1/\lambda^2\mu^2, \quad 1+2\epsilon'_{\theta\theta} = \mu^2(1+\psi^2\lambda^2r^2), \quad 1+2\epsilon'_{zz} = \lambda^2, \\ \epsilon'_{\theta z} = \psi\lambda^2\mu r, \quad \epsilon'_{zr} = 0 \quad \text{and} \quad \epsilon'_{r\theta} = 0. \end{aligned}\right\} \tag{4·2}$$

Substituting from (4·2) in the stress-stain relations (2·12) we have

$$\left.\begin{aligned} t_{\rho\rho} &= 2\left[\frac{1}{\lambda^2\mu^2}\frac{\partial W}{\partial I_1} - \lambda^2\mu^2\frac{\partial W}{\partial I_2} + I_2\frac{\partial W}{\partial I_2}\right]+p, \\ t_{\vartheta\vartheta} &= 2\left[\mu^2(1+\psi^2\lambda^2r^2)\frac{\partial W}{\partial I_1} - \frac{1}{\mu^2}\frac{\partial W}{\partial I_2} + I_2\frac{\partial W}{\partial I_2}\right]+p, \\ t_{\zeta\zeta} &= 2\left[\lambda^2\frac{\partial W}{\partial I_1} - \frac{1}{\lambda^2}(1+\psi^2\lambda^2r^2)\frac{\partial W}{\partial I_2} + I_2\frac{\partial W}{\partial I_2}\right]+p, \\ t_{\vartheta\zeta} &= 2\left[\psi\lambda^2\mu r\frac{\partial W}{\partial I_1} + \frac{\psi}{\mu}r\frac{\partial W}{\partial I_2}\right] \\ \text{and} \qquad t_{\zeta\rho} &= t_{\rho\vartheta} = 0, \end{aligned}\right\} \tag{4·3}$$

where p is the hydrostatic pressure, which must be determined from the equations of equilibrium. From (4·1) and (2·13) we find

$$I_1 = \lambda^2+\mu^2+\frac{1}{\lambda^2\mu^2}+\psi^2\lambda^2\mu^2r^2$$

and

$$I_2 = \frac{1}{\lambda^2}+\frac{1}{\mu^2}+\lambda^2\mu^2+\psi^2r^2. \tag{4·4}$$

The expressions for I_1 and I_2 and the stress components $t_{\rho\rho}$, etc., are seen to be functions of r only, provided that p is a function of r only, as it must be from considerations of symmetry. Substituting from (4·3) in equations (2·25), taking the body forces (R', Θ', Z') as zero, we see that the second and third equations are automatically satisfied and the first equation becomes

$$\frac{\partial t_{\rho\rho}}{\partial \rho}+\frac{t_{\rho\rho}-t_{\vartheta\vartheta}}{\rho} = 0. \tag{4·5}$$

We obtain

$$p = -2\int_{\mu_1 a_1}^{\rho}\frac{1}{\rho}\left\{\left[\frac{1}{\lambda^2\mu^2}-\mu^2(1+\psi^2\lambda^2r^2)\right]\frac{\partial W}{\partial I_1}-\left[\lambda^2\mu^2-\frac{1}{\mu^2}\right]\frac{\partial W}{\partial I_2}\right\}d\rho$$
$$-2\left[\frac{1}{\lambda^2\mu^2}\frac{\partial W}{\partial I_1}-\lambda^2\mu^2\frac{\partial W}{\partial I_2}+I_2\frac{\partial W}{\partial I_2}\right]+\kappa, \tag{4·6}$$

where κ is an integration constant.

Substituting for ρ, from (3·3), (4·6) becomes

$$p = -2\int_{a_1}^{r}\frac{1}{\lambda\mu^2r}\left\{\left[\frac{1}{\lambda^2\mu^2}-\mu^2(1+\psi^2\lambda^2r^2)\right]\frac{\partial W}{\partial I_1}-\left[\lambda^2\mu^2-\frac{1}{\mu^2}\right]\frac{\partial W}{\partial I_2}\right\}dr$$
$$-2\left[\frac{1}{\lambda^2\mu^2}\frac{\partial W}{\partial I_1}-\lambda^2\mu^2\frac{\partial W}{\partial I_2}+I_2\frac{\partial W}{\partial I_2}\right]+\kappa. \tag{4·7}$$

We have, employing the notation $R = r^2$,

$$\frac{dW}{dR} = \frac{\partial W}{\partial I_1}\frac{dI_1}{dR}+\frac{\partial W}{\partial I_2}\frac{dI_2}{dR}. \tag{4·8}$$

From (4·4), (3·2) and (3·3),

$$\frac{dI_1}{dR} = \frac{K}{\lambda\mu^2r^4}\left(\frac{1}{\lambda^2\mu^2}-\mu^2\right)+\psi^2\lambda$$

and

$$\frac{dI_2}{dR} = \frac{K}{\lambda\mu^2r^4}\left(\frac{1}{\mu^2}-\lambda^2\mu^2\right)+\psi^2. \tag{4·9}$$

Equations (4·7), (4·8) and (4·9) yield

$$p = -\frac{1}{K}\int_{a_1^2}^{R}R\frac{dW}{dR}dR+\frac{\psi^2}{K}\int_{a_1^2}^{R}R\left(\lambda^2\mu^2\frac{\partial W}{\partial I_1}+\frac{\partial W}{\partial I_2}\right)dR-2\left[\frac{1}{\lambda^2\mu^2}\frac{\partial W}{\partial I_1}-\lambda^2\mu^2\frac{\partial W}{\partial I_2}+I_2\frac{\partial W}{\partial I_2}\right]+\kappa. \tag{4·10}$$

5. The surface tractions

Let $R_{\nu1}$ and $R_{\nu2}$ denote the radial components of the surface tractions acting on the surfaces which are at $r = a_1$ and $r = a_2$ respectively in the undeformed state. These are measured per unit area of surface in the deformed state and are taken as positive when in the direction of the outward drawn normal to the surface considered, in its deformed state. Since, for the deformation considered, the stress components $t_{\rho\vartheta}$ and $t_{\zeta\rho}$ are both zero, the azimuthal and longitudinal components of the surface traction, on each of the curved surfaces, vanish.

Introducing into (2·15),

$$\cos(\rho,\nu') = 1 \quad \text{and} \quad \cos(\vartheta,\nu') = \cos(\zeta,\nu') = 0,$$

we have
$$R_{\nu 1} = (t_{\rho\rho})_{r=a_1} \quad \text{and} \quad R_{\nu 2} = (t_{\rho\rho})_{r=a_2}, \tag{5·1}$$

where $t_{\rho\rho}$ is given by (4·3). From (4·3) and (4·10),

$$t_{\rho\rho} = -\frac{1}{K}\int_{a_1^2}^{R} R\frac{dW}{dR}\,dR + \frac{\psi^2}{K}\int_{a_1^2}^{R} R\left(\lambda^2\mu^2\frac{\partial W}{\partial I_1}+\frac{\partial W}{\partial I_2}\right)dR + \kappa. \tag{5·2}$$

If $R_{\nu 1} = 0$, then, from (5·1) and (5·2),

$$\kappa = 0. \tag{5·3}$$

From (5·3) and (4·10), we obtain

$$p = -\frac{1}{K}\int_{a_1^2}^{R} R\frac{dW}{dR}\,dR + \frac{\psi^2}{K}\int_{a_1^2}^{R} R\left(\lambda^2\mu^2\frac{\partial W}{\partial I_1}+\frac{\partial W}{\partial I_2}\right)dR - 2\left[\frac{1}{\lambda^2\mu^2}\frac{\partial W}{\partial I_1} - \lambda^2\mu^2\frac{\partial W}{\partial I_2} + I_2\frac{\partial W}{\partial I_2}\right]. \tag{5·4}$$

If we also have $R_{\nu 2} = 0$, then

$$\int_{a_1^2}^{a_2^2} R\frac{dW}{dR}\,dR = \psi^2\int_{a_1^2}^{a_2^2} R\left(\lambda^2\mu^2\frac{\partial W}{\partial I_1}+\frac{\partial W}{\partial I_2}\right)dR. \tag{5·5}$$

Let $R_{\nu'}$, $\Theta_{\nu'}$ and $Z_{\nu'}$ denote the components of the surface traction, over a plane end of the tube, which are in the radial, azimuthal and longitudinal directions respectively in the deformed state and are measured per unit area of the surface in that state. Introducing into (2·15) the relations

$$\cos(\rho,\nu') = \cos(\vartheta,\nu') = 0 \quad \text{and} \quad \cos(\zeta,\nu') = 1,$$

and employing the expressions (4·3) and (5·4), we obtain

$$\left.\begin{aligned} R_{\nu'} &= 0, \quad \Theta_{\nu'} = 2\psi\lambda r\left(\lambda\mu\frac{\partial W}{\partial I_1}+\frac{1}{\lambda\mu}\frac{\partial W}{\partial I_2}\right) \\ \text{and} \qquad Z_{\nu'} &= 2\left[\left(\lambda^2-\frac{1}{\lambda^2\mu^2}\right)\frac{\partial W}{\partial I_1}+\left(\lambda^2\mu^2-\frac{1}{\lambda^2}-\psi^2r^2\right)\frac{\partial W}{\partial I_2}\right] \\ &\qquad -\frac{1}{K}\int_{a_1^2}^{R} R\frac{dW}{dR}\,dR + \frac{\psi^2}{K}\int_{a_1^2}^{R} R\left(\lambda^2\mu^2\frac{\partial W}{\partial I_1}+\frac{\partial W}{\partial I_2}\right)dR. \end{aligned}\right\} \tag{5·6}$$

The resultant couple M is given by

$$M = 2\pi\int_{\mu_2 a_2}^{\mu_1 a_1} \Theta_{\nu'}\rho^2\,d\rho = 4\pi\psi\int_{a_2}^{a_1} \mu r^3\left(\lambda\mu\frac{\partial W}{\partial I_1}+\frac{1}{\lambda\mu}\frac{\partial W}{\partial I_2}\right)dr, \tag{5·7}$$

and the resultant longitudinal force N is given by

$$\begin{aligned} N &= 2\pi\int_{\mu_2 a_2}^{\mu_1 a_1} Z_{\nu'}\rho\,d\rho = \frac{2\pi}{\lambda}\int_{a_2}^{a_1} Z_{\nu'}\,r\,dr \\ &= \frac{4\pi}{\lambda}\int_{a_2}^{a_1}\left[\left(\lambda^2-\frac{1}{\lambda^2\mu^2}\right)\frac{\partial W}{\partial I_1}+\left(\lambda^2\mu^2-\frac{1}{\lambda^2}-\psi^2r^2\right)\frac{\partial W}{\partial I_2}\right]r\,dr \\ &\qquad -\frac{\pi}{K\lambda}\int_{a_2^2}^{a_1^2}\int_{a_1^2}^{R} R\frac{dW}{dR}\,dR\,dR + \frac{\pi\psi^2}{K\lambda}\int_{a_2^2}^{a_1^2}\int_{a_1^2}^{R} R\left(\lambda^2\mu^2\frac{\partial W}{\partial I_1}+\frac{\partial W}{\partial I_2}\right)dR\,dR. \end{aligned} \tag{5·8}$$

Integrating the double integrals in (5·8) by parts, we obtain

$$N = \frac{4\pi}{\lambda}\int_{a_2}^{a_1}\left[\left(\lambda^2-\frac{1}{\lambda^2\mu^2}\right)\frac{\partial W}{\partial I_1}+\left(\lambda^2\mu^2-\frac{1}{\lambda^2}-\psi^2r^2\right)\frac{\partial W}{\partial I_2}\right]r\,dr-\frac{\pi}{K\lambda}a_2^2\int_{a_2^2}^{a_1^2}R\frac{dW}{dR}dR+\frac{\pi}{K\lambda}\int_{a_2^2}^{a_1^2}R^2\frac{dW}{dR}dR$$
$$+\frac{\pi\psi^2}{K\lambda}a_2^2\int_{a_2^2}^{a_1^2}R\left(\lambda^2\mu^2\frac{\partial W}{\partial I_1}+\frac{\partial W}{\partial I_2}\right)dR-\frac{\pi\psi^2}{K\lambda}\int_{a_2^2}^{a_1^2}R^2\left(\lambda^2\mu^2\frac{\partial W}{\partial I_1}+\frac{\partial W}{\partial I_2}\right)dR. \quad (5·9)$$

If $R_{\nu 2}=0$, we can employ the relation (5·5) in (5·9), yielding

$$N = \frac{4\pi}{\lambda}\int_{a_2}^{a_1}\left[\left(\lambda^2-\frac{1}{\lambda^2\mu^2}\right)\frac{\partial W}{\partial I_1}+\left(\lambda^2\mu^2-\frac{1}{\lambda^2}-\psi^2r^2\right)\frac{\partial W}{\partial I_2}\right]r\,dr$$
$$+\frac{\pi}{K\lambda}\int_{a_2^2}^{a_1^2}R^2\frac{dW}{dR}dR-\frac{\pi\psi^2}{K\lambda}\int_{a_2^2}^{a_1^2}R^2\left(\lambda^2\mu^2\frac{\partial W}{\partial I_1}+\frac{\partial W}{\partial I_2}\right)dR. \quad (5·10)$$

Introducing the equations (4·8), (4·9), (3·2) and (3·3) into (5·10), we obtain

$$N = \pi\int_{a_2^2}^{a_1^2}\left[\frac{2}{\lambda}\left(\lambda^2-\frac{1}{\lambda^2\mu^2}\right)\left(\frac{\partial W}{\partial I_1}+\mu^2\frac{\partial W}{\partial I_2}\right)+\frac{1}{\lambda^2\mu^2}\left(\frac{1}{\lambda^2\mu^2}-\mu^2\right)\left(\frac{\partial W}{\partial I_1}+\lambda^2\frac{\partial W}{\partial I_2}\right)-\psi^2R\left(\frac{\partial W}{\partial I_1}+\frac{2}{\lambda}\frac{\partial W}{\partial I_2}\right)\right]dR. \quad (5·11)$$

In the more general case, when $R_{\nu 2}\neq 0$, (5·9) yields, in a similar manner,

$$N = \pi\int_{a_2^2}^{a_1^2}\left[\frac{2}{\lambda}\left(\lambda^2-\frac{1}{\lambda^2\mu^2}\right)\left(\frac{\partial W}{\partial I_1}+\mu^2\frac{\partial W}{\partial I_2}\right)\right.$$
$$\left.+\frac{1}{\lambda^2\mu^2}\left(\frac{1}{\lambda^2\mu^2}-\mu^2\right)\left(\frac{\partial W}{\partial I_1}+\lambda^2\frac{\partial W}{\partial I_2}\right)\left(1-\frac{a_2^2}{R}\right)+\psi^2\frac{\partial W}{\partial I_1}(a_2^2-R)-2\frac{\psi^2}{\lambda}R\frac{\partial W}{\partial I_2}\right]dR. \quad (5·12)$$

Also, from (5·1), (5·2) and (5·3), we have

$$R_{\nu 2} = -\frac{1}{K}\int_{a_1^2}^{a_2^2}R\frac{dW}{dR}dR+\frac{\psi^2}{K}\int_{a_1^2}^{a_2^2}R\left(\lambda^2\mu^2\frac{\partial W}{\partial I_1}+\frac{\partial W}{\partial I_2}\right)dR. \quad (5·13)$$

6. The deformation of a cylindrical rod

For a cylindrical rod, we take $a_2=0$ in the formulae of the preceding section. Then, from (3·4), $K=0$ and, from (3·2) and (3·3), we have $\lambda\mu^2=1$. We obtain, from (5·6), (4·9) and (4·8),

$$\Theta_{\nu'} = 2\psi\lambda^{\frac{3}{2}}r\left(\frac{\partial W}{\partial I_1}+\frac{1}{\lambda}\frac{\partial W}{\partial I_2}\right)$$

and

$$Z_{\nu'} = 2\left[\left(\lambda^2-\frac{1}{\lambda}\right)\left(\frac{\partial W}{\partial I_1}+\frac{1}{\lambda}\frac{\partial W}{\partial I_2}\right)-\psi^2r^2\frac{\partial W}{\partial I_2}\right]+\lambda\psi^2\int_{a_1^2}^{R}\frac{\partial W}{\partial I_1}dR. \quad (6·1)$$

Equations (5·7) and (5·12) yield

$$M = 4\pi\psi\int_0^{a_1}r^3\left(\frac{\partial W}{\partial I_1}+\frac{1}{\lambda}\frac{\partial W}{\partial I_2}\right)dr$$

and

$$N = 4\pi\left(\lambda-\frac{1}{\lambda^2}\right)\int_0^{a_1}\left(\frac{\partial W}{\partial I_1}+\frac{1}{\lambda}\frac{\partial W}{\partial I_2}\right)r\,dr-2\pi\psi^2\int_0^{a_1}r^3\left(\frac{\partial W}{\partial I_1}+\frac{2}{\lambda}\frac{\partial W}{\partial I_2}\right)dr. \quad (6·2)$$

From (4·4), $$I_1=\lambda^2+\frac{2}{\lambda}+\lambda\psi^2r^2 \quad \text{and} \quad I_2=2\lambda+\frac{1}{\lambda^2}+\psi^2r^2. \quad (6·3)$$

From (6·3), $$\frac{dI_1}{dR} = \psi^2\lambda \quad\text{and}\quad \frac{dI_2}{dR} = \psi^2. \tag{6·4}$$

With (4·8), (6·4) gives $$\frac{dW}{dR} = \psi^2\lambda\left(\frac{\partial W}{\partial I_1} + \frac{1}{\lambda}\frac{\partial W}{\partial I_2}\right). \tag{6·5}$$

Thus, equations (6·2) become

$$\left.\begin{aligned} M &= \frac{2\pi}{\lambda\psi}\int_0^{a_1^2} R\frac{dW}{dR}\,dR = \frac{2\pi}{\lambda\psi}\left[a_1^2(W)_{r=a_1} - \int_0^{a_1^2} W\,dR\right] \\ \text{and}\quad N &= \frac{2\pi}{\psi^2}\left(1-\frac{1}{\lambda^3}\right)\int_0^{a_1^2}\frac{dW}{dR}\,dR - \frac{\pi}{\lambda}\int_0^{a_1^2} R\frac{dW}{dR}\,dR - \frac{\pi\psi^2}{\lambda}\int_0^{a_1^2} R\frac{\partial W}{\partial I_2}\,dR \\ &= \frac{2\pi}{\psi^2}\left(1-\frac{1}{\lambda^3}\right)[(W)_{r=a_1} - (W)_{r=0}] - \frac{\pi}{\lambda}\left[a_1^2(W)_{r=a_1} - \int_0^{a_1^2} W\,dR\right] - \frac{\pi\psi^2}{\lambda}\int_0^{a_1^2} R\frac{\partial W}{\partial I_2}\,dR. \end{aligned}\right\} \tag{6·6}$$

When $\lambda = 1$, i.e. the torsion is unaccompanied by extension, equations (6·2) yield

$$\left.\begin{aligned} M &= 4\pi\psi\int_0^{a_1} r^3\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right)dr \\ \text{and}\qquad N &= -2\pi\psi^2\int_0^{a_1} r^3\left(\frac{\partial W}{\partial I_1} + 2\frac{\partial W}{\partial I_2}\right)dr, \end{aligned}\right\} \tag{6·7}$$

in agreement with the results obtained, for this case, in a previous paper (Rivlin 1949*a*).

If $\psi = 0$, i.e. the torsion is zero, then M is, of course, zero and, from the second of equations (6·2),

$$N = 2\pi\left(\lambda - \frac{1}{\lambda^2}\right)\left(\frac{\partial W}{\partial I_1} + \frac{1}{\lambda}\frac{\partial W}{\partial I_2}\right)_{\psi=0} a_1^2. \tag{6·8}$$

From the first of equations (6·2),

$$(M/\psi)_{\psi=0} = \pi\left(\frac{\partial W}{\partial I_1} + \frac{1}{\lambda}\frac{\partial W}{\partial I_2}\right)_{\psi=0} a_1^4. \tag{6·9}$$

Equations (6·8) and (6·9) yield $$\frac{Na_1^2}{(M/\psi)_{\psi=0}} = 2\left(\lambda - \frac{1}{\lambda^2}\right). \tag{6·10}$$

This law which relates the force necessary to produce a large simple extension, with the torsional modulus for a small torsion superposed on that simple extension, is independent of the particular form of the stored-energy function which applies to the material.

If the stored-energy function has the particular form given by

$$W = C_1(I_1 - 3) + C_2(I_2 - 3), \tag{6·11}$$

then, from (6·1), we have

$$\left.\begin{aligned} \Theta_{\nu'} &= 2\psi\lambda^{\frac{3}{2}} r\left(C_1 + \frac{1}{\lambda}C_2\right) \\ \text{and}\qquad Z_{\nu'} &= 2\left(\lambda^2 - \frac{1}{\lambda}\right)\left(C_1 + \frac{1}{\lambda}C_2\right) + \psi^2[\lambda C_1(r^2 - a_1^2) - 2r^2C_2]. \end{aligned}\right\} \tag{6·12}$$

Again, from (6·6) or (6·2), we have

$$\left.\begin{aligned} M &= \pi\left(C_1 + \frac{1}{\lambda}C_2\right)\psi a_1^4 \\ \text{and}\qquad N &= 2\pi\left(\lambda - \frac{1}{\lambda^2}\right)\left(C_1 + \frac{1}{\lambda}C_2\right)a_1^2 - \tfrac{1}{2}\pi\left(C_1 + \frac{2}{\lambda}C_2\right)\psi^2 a_1^4. \end{aligned}\right\} \tag{6·13}$$

7. The tube turned inside out

For zero torsion, i.e. when $\psi = 0$, equation (5·5), which gives the condition that the surface tractions on both of the curved surfaces may vanish simultaneously, is

$$\int_{a_1^2}^{a_2^2} R\frac{dW}{dR}\,dR = 0. \tag{7·1}$$

Integrating by parts, this becomes

$$a_1^2(W)_{r=a_1} - a_2^2(W)_{r=a_2} = \int_{a_2^2}^{a_1^2} W\,dR. \tag{7·2}$$

I_1 and I_2 are given, from (4·4), by

$$I_1 = \lambda^2+\mu^2+\frac{1}{\lambda^2\mu^2} \quad \text{and} \quad I_2 = \frac{1}{\mu^2}+\frac{1}{\lambda^2}+\lambda^2\mu^2, \tag{7·3}$$

μ being given by (3·3) and (3·2).

Employing the relations (4·8) and (4·9), equation (7·1) may also be written in the form

$$\int_{a_2^2}^{a_1^2}\left[\frac{R}{(R+K)^2}-\frac{1}{R}\right]\left(\frac{\partial W}{\partial I_1}+\lambda^2\frac{\partial W}{\partial I_2}\right)dR = 0. \tag{7·4}$$

If $N = 0$, we have, from (5·11),

$$\int_{a_2^2}^{a_1^2}\left[\frac{2}{\lambda}\left(\lambda^2-\frac{1}{\lambda^2\mu^2}\right)\left(\frac{\partial W}{\partial I_1}+\mu^2\frac{\partial W}{\partial I_2}\right)+\frac{1}{\lambda^2\mu^2}\left(\frac{1}{\lambda^2\mu^2}-\mu^2\right)\left(\frac{\partial W}{\partial I_1}+\lambda^2\frac{\partial W}{\partial I_2}\right)\right]dR = 0. \tag{7·5}$$

Equations (7·5) and (7·2) or (7·4) are the basic equations for the determination of the extension ratio and changes in the radii of the tube when it is turned inside out and rests in a deformed state under the action of no external forces.

If the stored-energy function W for the material is given by

$$W = C_1(I_1-3)+C_2(I_2-3), \tag{7·6}$$

where C_1 and C_2 are physical constants characterizing the material, then equation (7·4) becomes

$$\int_{a_2^2}^{a_1^2}\left[\frac{R}{(R+K)^2}-\frac{1}{R}\right]dR = 0, \tag{7·7}$$

i.e.

$$2\log\frac{\mu_1}{\mu_2} = \frac{1}{\lambda}\left(\frac{1}{\mu_1^2}-\frac{1}{\mu_2^2}\right), \tag{7·8}$$

since K is given by (3·4).

Substituting for W from (7·6) in (7·5), we obtain

$$\int_{a_2^2}^{a_1^2}\left[\frac{2}{\lambda}\left(\lambda^2-\frac{1}{\lambda^2\mu^2}\right)(C_1+\mu^2C_2)+\frac{1}{\lambda^2\mu^2}\left(\frac{1}{\lambda^2\mu^2}-\mu^2\right)(C_1+\lambda^2C_2)\right]dR = 0. \tag{7·9}$$

Introducing into (7·9) the expression for μ given by (3·2) and (3·3), and carrying out the integration, we obtain

$$\frac{2}{\lambda}\left(\lambda^2-\frac{1}{\lambda}\right)\left(C_1+\frac{1}{\lambda}C_2\right)(a_1^2-a_2^2)+4KC_2\log\frac{\mu_2}{\mu_1}+\frac{K}{\lambda^3}(C_1+\lambda^2C_2)\left(\frac{1}{\mu_1^2}-\frac{1}{\mu_2^2}\right) = 0. \tag{7·10}$$

Equations (7·8) and (7·10), together with (3·4), can be used to determine λ, μ_1 and μ_2, giving the dimensional changes in the tube when it is turned inside out. Writing

$$-\lambda\mu_1^2 = \epsilon_1, \quad -\lambda\mu_2^2 = \epsilon_2 \quad \text{and} \quad a_1^2/a_2^2 = \chi, \tag{7·11}$$

equations (7·8) and (3·4) become

$$\log\frac{\epsilon_1}{\epsilon_2} = \frac{1}{\epsilon_2} - \frac{1}{\epsilon_1} \quad \text{and} \quad \chi(\epsilon_1+1) = \epsilon_2+1. \tag{7·12}$$

Since equations (7·12) are independent of C_1 and C_2, the ratio C_2/C_1 does not affect the values of ϵ_1 and ϵ_2. It is readily seen that ϵ_1 is the ratio of the volumes contained by the surface initially at $r = a_1$ in its deformed and undeformed states, and ϵ_2 is the corresponding ratio for the surface initially at $r = a_2$.

Equations (7·12) may be conveniently solved graphically by plotting ϵ_1 against ϵ_2 from the first of the equations and finding the intersection of the curve so obtained with the straight line, given by the second of the equations, of which the slope and axial intercepts depend on χ.

Corresponding values of ϵ_1 and ϵ_2 in the first of equations (7·12) are obtained by considering the equation

$$\varpi = \log\epsilon + \frac{1}{\epsilon}. \tag{7·13}$$

As ϵ increases from 0 to 1, ϖ falls monotonically from ∞ to 1 and, as ϵ increases from 1 to ∞, ϖ increases monotonically from 1 to ∞. Thus, to any value of ϖ greater than 1, there correspond two values of ϵ. These are corresponding values of ϵ_1 and ϵ_2 in the first of equations (7·12). Thus, apart from the straight line $\epsilon_1 = \epsilon_2$, the $\epsilon_1 - \epsilon_2$ graph consists of a curve symmetrical about $\epsilon_1 = \epsilon_2$, on which each value of ϵ_1 determines uniquely a value of ϵ_2 and conversely. We are only concerned with the part of the $\epsilon_1 - \epsilon_2$ curve, as shown in figure 1, for which $\epsilon_2 > \epsilon_1$, since only such values can satisfy the second of equations (7·12). It is apparent that $\epsilon_2 > 1$ and $\epsilon_1 < 1$. If $\epsilon_1 = \epsilon_2$, it can be seen from the second of equations (7·12) that since $\chi > 1$, $\epsilon_1 = \epsilon_2 = -1$, i.e. the tube is undeformed.

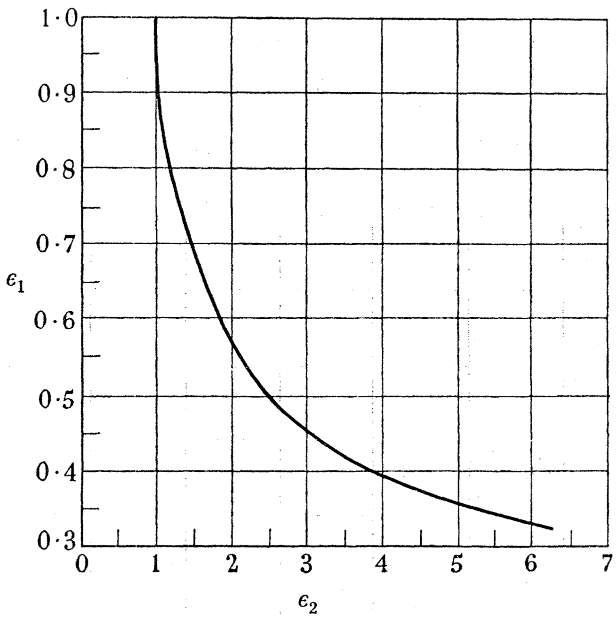

FIGURE 1

With the notation of (7·11), (7·10) may be written as

$$\frac{2}{\lambda}\left(\lambda^2 - \frac{1}{\lambda}\right)\left(C_1 + \frac{1}{\lambda}C_2\right)(\chi-1) - 2(\epsilon_2+1)\,C_2\log\frac{\epsilon_2}{\epsilon_1} + \frac{1}{\lambda^2}(\epsilon_2+1)\,(C_1+\lambda^2 C_2)\left(\frac{1}{\epsilon_1} - \frac{1}{\epsilon_2}\right) = 0, \tag{7·14}$$

which, with (7·12), yields

$$\frac{2(\lambda^3-1)(\lambda C_1+C_2)(\chi-1)}{\lambda(C_1-\lambda^2C_2)} = (1+\epsilon_2)\left(\frac{1}{\epsilon_2}-\frac{1}{\epsilon_1}\right). \qquad (7\cdot15)$$

If χ is known, then ϵ_1 and ϵ_2 may be determined from equation (7·12), and a measurement of λ would enable us to find C_2/C_1 from (7·15).

8. Simultaneous torsion and extension of a tube

If no force is applied to the inner curved surface of the tube and its stored-energy function is given by (6·11), we have, from (5·5),

$$\left[2\log\frac{\mu_1}{\mu_2}+\frac{1}{\lambda}\left(\frac{1}{\mu_2^2}-\frac{1}{\mu_1^2}\right)\right](C_1+\lambda^2C_2)+C_1\psi^2\lambda^2(a_2^2-a_1^2) = 0. \qquad (8\cdot1)$$

Employing the notation

$$\lambda\mu_1^2 = \epsilon_1, \quad \lambda\mu_2^2 = \epsilon_2, \quad \frac{a_1^2}{a_2^2} = \chi \quad \text{and} \quad \frac{C_1\psi^2\lambda^2(a_1^2-a_2^2)}{C_1+\lambda^2C_2} = \gamma_0, \qquad (8\cdot2)$$

(8·1) becomes

$$\left(\log\epsilon_1-\frac{1}{\epsilon_1}\right)-\left(\log\epsilon_2-\frac{1}{\epsilon_2}\right) = \gamma_0. \qquad (8\cdot3)$$

We also have the relation (3·4), which, with the notation of (8·2), may be written

$$\chi(\epsilon_1-1) = \epsilon_2-1. \qquad (8\cdot4)$$

For given values of χ and γ_0, ϵ_1 and ϵ_2 may be found by solving (8·3) and (8·4).

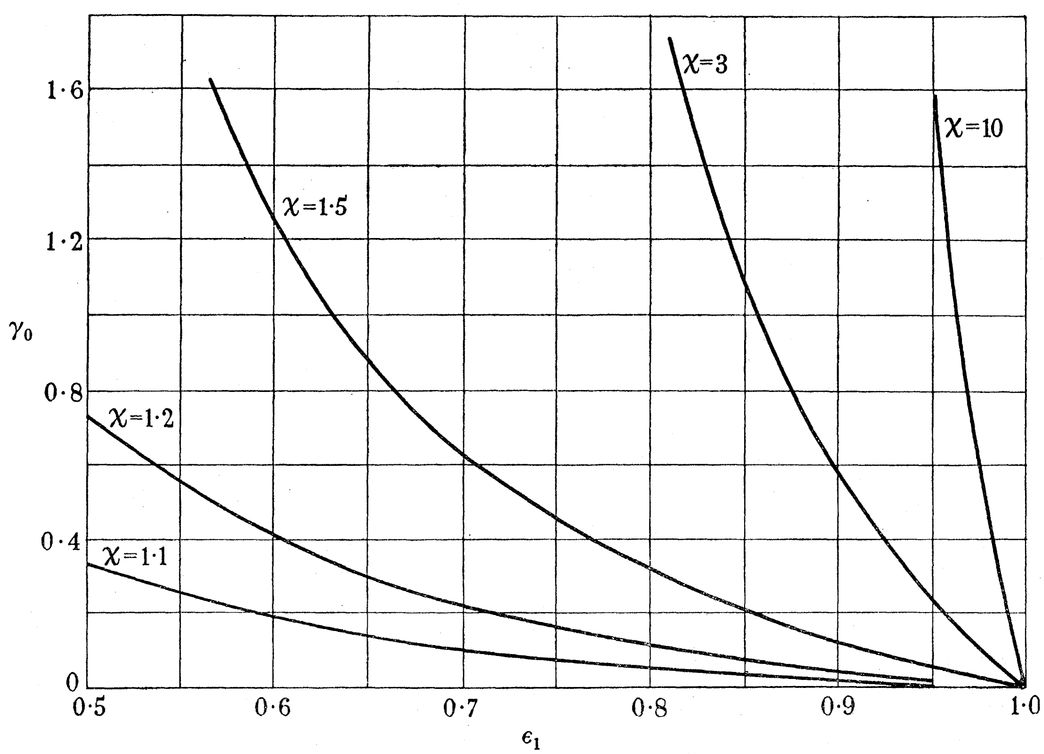

Figure 2

It will be assumed that ϵ_1 and ϵ_2 are both positive, i.e. the tube is not turned inside out. It is seen from (8·4) that either (ϵ_1-1) and (ϵ_2-1) are both positive or both negative. Since, by definition, $\chi>1$, in the former case $\epsilon_1<\epsilon_2$ and in the latter case $\epsilon_1>\epsilon_2$.

If, now, we assume C_1 and C_2 both positive, γ_0 is positive. It is noted that $(\log\epsilon-1/\epsilon)$ is a monotonic increasing function of ϵ, for ϵ positive. Consequently, in (8·3), $\epsilon_1>\epsilon_2$ and hence (ϵ_1-1) and (ϵ_2-1) are both negative.

For any value of χ, corresponding values of ϵ_1 and ϵ_2 can be obtained from (8·4). These may be substituted in (8·3), to give corresponding values of γ_0 and γ_0 against ϵ_1 curves obtained for various values of χ as shown in figure 2.

SIMULTANEOUS EXTENSION, INFLATION AND SHEAR OF A CYLINDRICAL ANNULUS

9. Definition of the deformation

In this part of the paper, we shall consider the body described in § 3 to be subjected to the following successive deformations:

(i) a uniform simple extension of extension ratio λ;

(ii) a uniform inflation of the tube in which its length remains constant and its external and internal radii change to $\mu_1 a_1$ and $\mu_2 a_2$ respectively;

(iii) a simple shear of the tube about its axis, in which each point moves about the axis through an angle ϕ which is dependent only on the radial position of the point;

(iv) a simple shear of the tube, in which each point moves parallel to the axis of the tube through a distance w which depends only on the radial position of the point.

In the deformations (i) and (ii), a point initially at (r,θ,z) moves to $(\rho,\theta,\lambda z)$, where ρ is a function of r only. In the deformation (iii), this point moves to $(\rho,\theta+\phi,\lambda z)$ and in the deformation (iv) to $(\rho,\theta+\phi,\lambda z+w)$, where ϕ and w are functions of r only. Thus the co-ordinates (ρ,ϑ,ζ), in the deformed state, of a point, which in the undeformed state is at (r,θ,z), may be written as

$$\rho=\mu r=[(r^2+K)/\lambda]^{\frac{1}{2}},\quad \vartheta=\theta+\phi\quad\text{and}\quad \zeta=\lambda z+w, \tag{9·1}$$

where, as in § 3,

$$K=a_1^2(\lambda\mu_1^2-1)=a_2^2(\lambda\mu_2^2-1). \tag{9·2}$$

10. The stress-strain relations and equations of equilibrium

Since, in the deformation considered, a point which is initially at (r,θ,z) moves to (ρ,ϑ,ζ), given by (9·1), we have, from (2·9),

$$\left.\begin{aligned} a_{11}&=1/\mu\lambda, & a_{12}&=0, & a_{13}&=0,\\ a_{21}&=\mu r\phi_r, & a_{22}&=\mu, & a_{23}&=0,\\ a_{31}&=w_r, & a_{32}&=0, & a_{33}&=\lambda.\end{aligned}\right\} \tag{10·1}$$

Employing (10·1) in (2·10), we obtain

$$\left.\begin{aligned} 1+2\epsilon'_{rr}&=1/\mu^2\lambda^2, & 1+2\epsilon'_{\theta\theta}&=\mu^2(1+r^2\phi_r^2), & 1+2\epsilon'_{zz}&=\lambda^2+w_r^2,\\ \epsilon'_{\theta z}&=\mu r\phi_r w_r, & \epsilon'_{zr}&=w_r/\mu\lambda, & \epsilon'_{r\theta}&=r\phi_r/\lambda.\end{aligned}\right\} \tag{10·2}$$

With (10·2), the stress-strain relations (2·12) yield

$$\left.\begin{aligned}
t_{\rho\rho} &= 2\left[\frac{1}{\mu^2\lambda^2}\frac{\partial W}{\partial I_1} - \mu^2(\lambda^2 + w_r^2 + \lambda^2 r^2\phi_r^2)\frac{\partial W}{\partial I_2} + I_2\frac{\partial W}{\partial I_2}\right] + p,\\
t_{\vartheta\vartheta} &= 2\left[\mu^2(1 + r^2\phi_r^2)\frac{\partial W}{\partial I_1} - \frac{1}{\mu^2}\frac{\partial W}{\partial I_2} + I_2\frac{\partial W}{\partial I_2}\right] + p,\\
t_{\zeta\zeta} &= 2\left[(\lambda^2 + w_r^2)\frac{\partial W}{\partial I_1} - \frac{1}{\lambda^2}\frac{\partial W}{\partial I_2} + I_2\frac{\partial W}{\partial I_2}\right] + p,\\
t_{\vartheta\zeta} &= 2\mu r\phi_r w_r\frac{\partial W}{\partial I_1},\\
t_{\zeta\rho} &= 2\frac{w_r}{\lambda}\left(\frac{1}{\mu}\frac{\partial W}{\partial I_1} + \mu\frac{\partial W}{\partial I_2}\right)\\
\text{and}\quad t_{\rho\vartheta} &= 2\frac{r\phi_r}{\lambda}\left(\frac{\partial W}{\partial I_1} + \lambda^2\frac{\partial W}{\partial I_2}\right).
\end{aligned}\right\} \qquad (10·3)$$

From (10·1) and (2·13), we see that

$$\left.\begin{aligned}
I_1 &= \mu^2 + \lambda^2 + \frac{1}{\mu^2\lambda^2} + \mu^2 r^2\phi_r^2 + w_r^2\\
\text{and}\quad I_2 &= \frac{1}{\mu^2} + \frac{1}{\lambda^2} + \mu^2\lambda^2 + \mu^2\lambda^2 r^2\phi_r^2 + \mu^2 w_r^2.
\end{aligned}\right\} \qquad (10·4)$$

Since $t_{\rho\rho}$, $t_{\vartheta\vartheta}$, ..., $t_{\rho\vartheta}$, given by equation (10·3), depend only on r, i.e. on ρ, the equations of equilibrium (2·25), in the absence of body forces, become

$$\left.\begin{aligned}
\frac{\partial t_{\rho\rho}}{\partial\rho} + \frac{t_{\rho\rho} - t_{\vartheta\vartheta}}{\rho} &= 0,\\
\frac{\partial t_{\rho\vartheta}}{\partial\rho} + \frac{2}{\rho}t_{\rho\vartheta} &= 0\\
\text{and}\quad \frac{\partial t_{\zeta\rho}}{\partial\rho} + \frac{1}{\rho}t_{\zeta\rho} &= 0.
\end{aligned}\right\} \qquad (10·5)$$

The second and third equations yield directly

$$t_{\rho\vartheta}\rho^2 = 2A/\lambda^2 \quad \text{and} \quad t_{\zeta\rho}\rho = 2B/\lambda^2, \qquad (10·6)$$

where A and B are constants of integration and ρ is given by (9·1). Equations (10·6) and (10·3) yield

$$\left.\begin{aligned}
\phi_r &= \frac{A}{r(r^2+K)\left(\dfrac{\partial W}{\partial I_1} + \lambda^2\dfrac{\partial W}{\partial I_2}\right)}\\
\text{and}\quad w_r &= \frac{Br}{\lambda r^2\dfrac{\partial W}{\partial I_1} + (r^2+K)\dfrac{\partial W}{\partial I_2}}.
\end{aligned}\right\} \qquad (10·7)$$

It is seen, from (10·4), that I_1 and I_2 depend on ϕ_r and w_r, and since, in general, $\partial W/\partial I_1$ and $\partial W/\partial I_2$ are functions of I_1 and I_2, equations (10·7) are non-linear differential equations for the determination of ϕ and w as functions of r. If, however, W is given by

$$W = C_1(I_1 - 3) + C_2(I_2 - 3), \qquad (10·8)$$

where C_1 and C_2 are constants, then equations (10·7) become

$$\phi_r = \frac{A}{r(r^2+K)(C_1+\lambda^2C_2)} \quad \text{and} \quad w_r = \frac{Br}{\lambda r^2C_1+(r^2+K)C_2}. \tag{10·9}$$

Integrating equations (10·9), we obtain

$$\phi = \frac{A}{K(C_1+\lambda^2C_2)}\log\frac{\mu_2}{\mu}+A',$$

and

$$w = \frac{B}{2(\lambda C_1+C_2)}\log\frac{(C_1+\mu^2C_2)\,r^2}{(C_1+\mu_2^2C_2)\,a_2^2}+B'. \tag{10·10}$$

Now, let us suppose that $w=0$ and $\phi=0$, when $r=a_2$, and that $w=w_0$ and $\phi=\phi_0$, when $r=a_1$. Employing these conditions to evaluate the integration constants A, B, A' and B', we obtain

$$\left.\begin{aligned} A &= \phi_0K(C_1+\lambda^2C_2)/\log\frac{\mu_2}{\mu_1} \\ \text{and} \qquad B &= 2w_0(\lambda C_1+C_2)/\log\frac{(C_1+\mu_1^2C_2)\,a_1^2}{(C_1+\mu_2^2C_2)\,a_2^2} \\ \text{and} \qquad A' &= B' = 0. \end{aligned}\right\} \tag{10·11}$$

From (10·11) and (10·10)

$$\phi = \phi_0'\log\frac{\mu_2}{\mu} \quad \text{and} \quad w = w_0'\log\frac{(C_1+\mu^2C_2)\,r^2}{(C_1+\mu_2^2C_2)\,a_2^2}, \tag{10·12}$$

where

$$\phi_0' = \phi_0/\log\frac{\mu_2}{\mu_1} \quad \text{and} \quad w_0' = w_0\bigg/\frac{(C_1+\mu_1^2C_2)\,a_1^2}{(C_1+\mu_2^2C_2)\,a_2^2}. \tag{10·13}$$

Substituting from (10·11) and (10·13) in (10·9), we obtain

$$\phi_r = \frac{\phi_0'K}{\lambda\mu^2r^3} \quad \text{and} \quad w_r = \frac{2(C_1\lambda+C_2)\,w_0'}{\lambda(C_1+\mu^2C_2)\,r}. \tag{10·14}$$

The first of equations (10·5) yields, with (10·3) and (10·8),

$$\frac{dp}{d\rho} = \frac{2}{\rho}\left[\left(\mu^2+\mu^2r^2\phi_r^2-\frac{1}{\mu^2\lambda^2}\right)(C_1+\lambda^2C_2)+\mu^2w_r^2C_2\right]-2\frac{d}{d\rho}\left[\frac{1}{\mu^2\lambda^2}C_1+\left(\frac{1}{\mu^2}+\frac{1}{\lambda^2}\right)C_2\right], \tag{10·15}$$

where ϕ_r and w_r are, of course, given by (10·14). This yields

$$p = \int^r \frac{2}{\lambda\mu^2r}\left[\left(\mu^2+\mu^2r^2\phi_r^2-\frac{1}{\mu^2\lambda^2}\right)(C_1+\lambda^2C_2)+\mu^2w_r^2C_2\right]dr-2\left[\frac{1}{\mu^2\lambda^2}C_1+\left(\frac{1}{\mu^2}+\frac{1}{\lambda^2}\right)C_2\right]. \tag{10·16}$$

Introducing into (10·16) the expressions (10·14) for ϕ_r and w_r and carrying out the integration, we obtain

$$\begin{aligned} p = {} & -\frac{1}{\lambda}(C_1+\lambda^2C_2)\left[\left(2\log\mu-\frac{1}{\lambda\mu^2}\right)+\phi_0'^2\left(2\log\mu+\frac{1}{\lambda\mu^2}\right)\right] \\ & -\frac{4}{\lambda^2}C_2w_0'^2\frac{C_1\lambda+C_2}{(C_1+\mu^2C_2)\,r^2}-2\left[\frac{1}{\mu^2\lambda^2}C_1+\left(\frac{1}{\mu^2}+\frac{1}{\lambda^2}\right)C_2\right]+\kappa, \end{aligned} \tag{10·17}$$

where κ is an integration constant.

11. The surface tractions

Let R_1', Θ_1', Z_1' denote the components of the surface traction acting on the outer curved surface of the body, at $r = a_1$ in the undeformed state, which are radial, azimuthal and longitudinal respectively in the deformed state of the body and are measured per unit area of the deformed surface. Let R_2', Θ_2', Z_2' denote the components of the surface traction acting on the inner curved surface of the body at $r = a_2$ in the undeformed state, and defined in an analogous manner. Then, from (2·15),

$$R_1' = (t_{\rho\rho})_{r=a_1}, \quad \Theta_1' = (t_{\rho\vartheta})_{r=a_1} \quad \text{and} \quad Z_1' = (t_{\zeta\rho})_{r=a_1}, \tag{11·1}$$

and

$$R_2' = (t_{\rho\rho})_{r=a_2}, \quad \Theta_2' = (t_{\rho\vartheta})_{r=a_2} \quad \text{and} \quad Z_2' = (t_{\zeta\rho})_{r=a_2}. \tag{11·2}$$

Employing the relations (10·14) and (10·8) in (10·3) and the resulting expressions for $t_{\rho\rho}$, $t_{\rho\vartheta}$ and $t_{\zeta\rho}$ in (11·1), we obtain

$$\left.\begin{aligned} R_1' &= \frac{2}{\lambda^2\mu_1^2}[C_1 + (\lambda^2+\mu_1^2)\,C_2] + p_1, \\ \Theta_1' &= \frac{2\phi_0' K}{\lambda^2\mu_1^2 a_1^2}(C_1+\lambda^2 C_2) \\ \text{and} \qquad Z_1' &= \frac{4w_0'(C_1\lambda + C_2)}{\lambda^2\mu_1 a_1}, \end{aligned}\right\} \tag{11·3}$$

where p_1 is given, from (10·17), by writing $\mu = \mu_1$ and $r = a_1$. R_2', Θ_2' and Z_2' are given by expressions similar to (11·3), μ_1 and a_1 being replaced by μ_2 and a_2.

Again, let R_ν', Θ_ν' and Z_ν' denote the components of the surface traction, acting on the initially plane ends of the body, which are radial, azimuthal and longitudinal respectively in the deformed state of the body and are measured per unit area of the surface in its undeformed state. These can be calculated from equations (2·16) and (2·18) to (2·24) by introducing into them the expressions (10·1) and

$$\cos(\rho,\nu) = \cos(\vartheta,\nu) = 0 \quad \text{and} \quad \cos(\zeta,\nu) = \pm 1. \tag{11·4}$$

We thus obtain

$$\left.\begin{aligned} R_\nu' &= -w_r\left(\frac{2}{\mu}C_2 + \mu p\right), \\ \Theta_\nu' &= -2\mu r\lambda\phi_r w_r C_2 \\ \text{and} \qquad Z_\nu' &= 2\lambda C_1 + 2\lambda\left(\frac{1}{\mu^2\lambda^2} + \mu^2 + \mu^2 r^2\phi_r^2\right)C_2 + \frac{p}{\lambda}, \end{aligned}\right\} \tag{11·5}$$

where p is given by (10·17) and ϕ_r and w_r by (10·14).

12. The applied forces

The forces which must be applied to the curved surfaces of the tube are:

(i) normal surface tractions R_1 and R_2, per unit length of tube measured in its undeformed state, acting respectively on the surfaces which are initially at $r = a_1$ and $r = a_2$, in the directions of the outward drawn normals to these surfaces;

(ii) an axial couple of magnitude M per unit length of the undeformed tube, acting on each curved surface;

(iii) a longitudinal force L, per unit length of the undeformed tube, acting on each curved surface.

These are given, from (11·3), by

$$\left.\begin{aligned} R_1 &= 2\pi a_1\left\{\frac{2}{\lambda\mu_1}\left[C_1+(\mu_1^2+\lambda^2)\,C_2\right]+p_1\lambda\mu_1\right\},\\ R_2 &= 2\pi a_2\left\{\frac{2}{\lambda\mu_2}\left[C_1+(\mu_2^2+\lambda^2)\,C_2\right]+p_2\lambda\mu_2\right\},\\ M &= 2\pi\lambda\mu_1^2a_1^2\,\Theta_1' = 4\pi(C_1+\lambda^2C_2)\,K\phi_0'/\lambda\\ \text{and}\qquad L &= 8\pi w_0'(C_1+C_2/\lambda). \end{aligned}\right\} \tag{12·1}$$

The surface traction (11·5), which is distributed over the free ends of the annulus, should also be applied in order to produce the deformation considered. If, in its deformed state, the length of the tube is sufficiently great compared with its thickness $(\mu_1 a_1-\mu_2 a_2)$ so that Saint-Venant's principle may be applied, the exact distribution, given by (11·5), of the surface forces over the ends of the tube, is not important, provided that a statically equivalent system of forces is applied at or near the ends. If the highly elastic material is deformed by applying forces to rigid cylindrical shells, which are in contact with the curved surfaces of the body and cannot move relative to them, then this statically equivalent system can also be applied by means of these cylindrical shells and will make a contribution to the forces acting on the rigid shells near their ends. However, their resultant effect on the force L and couple M which must be applied to the rigid cylindrical shells is zero, since they have equal magnitudes and opposite sign at each end of such a shell.

It should be noted that throughout the analysis the hydrostatic pressure p—and therefore the forces R_1 and R_2, given by (12·1)—are undetermined to the extent of an arbitrary constant. The value of the constant cannot be determined unless the conditions actually obtaining at the free ends are investigated in detail. Its value does not, however, affect the values of the couple M and tangential force L given by (12·1).

13. The case of no extension or inflation

If there is no extension or inflation, then $\mu_1=\mu_2=\lambda=1$. From (9·2) we obtain $K=0$. In this case, the integration of equations (10·9) does not lead to (10·10). Equations (10·9) become

$$\phi_r = \frac{A}{(C_1+C_2)\,r^3}\quad\text{and}\quad w_r=\frac{B}{(C_1+C_2)\,r}. \tag{13·1}$$

Integrating these equations, we obtain

$$\phi = -\frac{A}{2(C_1+C_2)\,r^2}+A'\quad\text{and}\quad w=\frac{B}{C_1+C_2}\log r+B'. \tag{13·2}$$

Taking $w=0$ and $\phi=0$ when $r=a_2$, and $w=w_0$ and $\phi=\phi_0$ when $r=a_1$, as before, we obtain

$$\left.\begin{aligned} &A = 2\phi_0(C_1+C_2)\,a_1^2a_2^2/(a_1^2-a_2^2)\quad\text{and}\quad B=w_0(C_1+C_2)/\log\frac{a_1}{a_2}\\ \text{and}\qquad &A' = \phi_0 a_1^2/(a_1^2-a_2^2)\quad\text{and}\quad B'=-w_0\log a_2/\log\frac{a_1}{a_2}. \end{aligned}\right\} \tag{13·3}$$

Introducing (13·3) into (13·2), we obtain

$$\phi = \frac{\phi_0 a_1^2}{a_1^2-a_2^2}\left(1-\frac{a_2^2}{r^2}\right)\quad\text{and}\quad w=w_0\log\frac{r}{a_2}\Big/\log\frac{a_1}{a_2}. \tag{13·4}$$

Since $K = 0$, we see, from (9·1), that $\mu = 1$. Introducing $\mu = \lambda = 1$ into (10·16), we obtain

$$p = -\int^r \frac{2}{r}[r^2\phi_r^2(C_1+C_2)+w_r^2 C_2]\,dr. \tag{13·5}$$

Introducing into (13·5) the expressions (13·1) for ϕ_r and w_r, we obtain

$$\begin{aligned} p &= -\int^r \frac{2}{(C_1+C_2)^2 r^5}[(C_1+C_2)A^2+C_2B^2r^2]\,dr \\ &= \frac{A^2}{2(C_1+C_2)r^4}+\frac{C_2B^2}{(C_1+C_2)^2r^2}+\kappa, \end{aligned} \tag{13·6}$$

where κ is an integration constant.

The surface tractions on the curved surfaces are given by equations (11·1), (11·2), (10·3) and (10·8). Those on the plane ends are given by (11·5). In all of these, we write $\mu = \mu_1 = \mu_2 = \lambda = 1$. ϕ_r and w_r are given by (13·1) and p by (13·6). We obtain

$$R_1' = 2(C_1+2C_2)+p_1, \quad \Theta_1' = 2A/a_1^2 \quad \text{and} \quad Z_1' = 2B/a_1, \tag{13·7}$$

where p_1 is given from (13·6) by writing $r = a_1$. R_2', Θ_2' and Z_2' are given by expressions similar to (13·7) in which a_2 is substituted for a_1.

The resultant forces R_1, R_2, M and L, defined as in § 12, are given, from (13·7), by

$$R_1 = 2\pi a_1(2C_1+4C_2+p_1), \quad R_2 = 2\pi a_2(2C_1+4C_2+p_2), \quad M = 4\pi A \quad \text{and} \quad L = 4\pi B, \tag{13·8}$$

where p_1 and p_2 are given by introducing $r = a_1$ and a_2 respectively into (13·6).

THE FLEXURE OF A CUBOID

14. The definition of the deformation

In a previous paper (Rivlin 1949*b*) the simple flexure of a cuboid of incompressible, isotropic, highly elastic material was considered. However, the discussion was specialized at an early stage to the case when the stored-energy function has the form (6·11). It is possible to carry the discussion further than was done there without specifying the form of W as a function of I_1 and I_2. Here, the further generalization obtained by considering a uniform extension normal to the plane of flexure to be superposed on the simple flexure is also made.

It is considered that initially the cuboid has edges of length $2a$, $2b$ and $2c$ parallel to the axes of a rectangular Cartesian co-ordinate system (x, y, z), and that it is deformed symmetrically with respect to the x-axis so that:

(i) each plane initially normal to the x-axis becomes, in the deformed state, a portion of the curved surface of a cylinder having the z-axis as axis;

(ii) planes initially normal to the y-axis become in the deformed state planes containing the z-axis;

(iii) the displacement parallel to the z-axis of a point initially at (x, y, z) is $(\lambda-1)z$, where λ is a constant.

If we choose a cylindrical polar co-ordinate system (r, θ, z), so that

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta \tag{14·1}$$

and consider that the point initially at (r, θ, z) moves, in the deformation, to (ρ, ϑ, ζ), then it can be shown in a manner similar to that employed in the previous paper (Rivlin 1949 *b*) that

$$\rho = (2Ax+B)^{\frac{1}{2}}, \quad \vartheta = y/\lambda A \quad \text{and} \quad \zeta = \lambda z, \tag{14·2}$$

where
$$A = \frac{r_1^2 - r_2^2}{4a} \quad \text{and} \quad B = \frac{a_1 r_2^2 - a_2 r_1^2}{2a}. \tag{14·3}$$

r_1 and r_2 $(r_1 > r_2)$ are the radii of the curved surfaces of the deformed body, which are initially the planes $x = a_1$ and $x = a_2$ respectively, so that $a_1 - a_2 = 2a$.

15. The stress-strain relations and equations of equilibrium

By substituting from (14·1) and (14·2) in the equations of § 2, we obtain

$$\left.\begin{aligned} t_{\rho\rho} &= 2\left(\frac{A^2}{\rho^2}\frac{\partial W}{\partial I_1} - \frac{\rho^2}{A^2}\frac{\partial W}{\partial I_2} + I_2\frac{\partial W}{\partial I_2}\right) + p, \\ t_{\vartheta\vartheta} &= 2\left(\frac{\rho^2}{\lambda^2 A^2}\frac{\partial W}{\partial I_1} - \frac{\lambda^2 A^2}{\rho^2}\frac{\partial W}{\partial I_2} + I_2\frac{\partial W}{\partial I_2}\right) + p, \\ t_{\zeta\zeta} &= 2\left(\lambda^2\frac{\partial W}{\partial I_1} - \frac{1}{\lambda^2}\frac{\partial W}{\partial I_2} + I_2\frac{\partial W}{\partial I_2}\right) + p \\ \text{and} \quad t_{\vartheta\zeta} &= t_{\zeta\rho} = t_{\rho\vartheta} = 0, \end{aligned}\right\} \tag{15·1}$$

where
$$I_1 = \frac{A^2}{\rho^2} + \frac{\rho^2}{\lambda^2 A^2} + \lambda^2 \quad \text{and} \quad I_2 = \frac{\rho^2}{A^2} + \frac{\lambda^2 A^2}{\rho^2} + \frac{1}{\lambda^2}. \tag{15·2}$$

Employing (15·1) in (2·25), we have, in the absence of body forces,

$$\begin{aligned} p &= -2\left(\frac{A^2}{\rho^2}\frac{\partial W}{\partial I_1} - \frac{\rho^2}{A^2}\frac{\partial W}{\partial I_2} + I_2\frac{\partial W}{\partial I_2}\right) + \int^{\rho}\frac{2}{\rho}\left(\frac{\rho^2}{\lambda^2 A^2} - \frac{A^2}{\rho^2}\right)\left(\frac{\partial W}{\partial I_1} + \lambda^2\frac{\partial W}{\partial I_2}\right)d\rho \\ &= -2\left(\frac{A^2}{\rho^2}\frac{\partial W}{\partial I_1} - \frac{\rho^2}{A^2}\frac{\partial W}{\partial I_2} + I_2\frac{\partial W}{\partial I_2}\right) + W + \kappa, \end{aligned} \tag{15·3}$$

where κ is an integration constant.

16. The surface tractions

The surface tractions which must be applied to the surfaces of the body in order to produce the deformation described by (14·2) are obtained from (15·1) and (2·15). We find that the surface tractions are normal to the surfaces to which they are applied, in their deformed state. We shall denote those applied to the curved surfaces $\rho = r_1$ and $\rho = r_2$ by R_1' and R_2', those applied to the plane surfaces which pass through the z-axis by Θ' and those applied to the surfaces normal to the z-axis by Z'. R_1', R_2', Θ' and Z' are measured per unit area of the surfaces in their deformed states. They are given by

$$\left.\begin{aligned} R_1' &= (W)_{\rho=r_1} + \kappa, \quad R_2' = (W)_{\rho=r_2} + \kappa, \\ \Theta' &= 2\left(\frac{\rho^2}{\lambda^2 A^2} - \frac{A^2}{\rho^2}\right)\left(\frac{\partial W}{\partial I_1} + \lambda^2\frac{\partial W}{\partial I_2}\right) + W + \kappa \\ \text{and} \quad Z' &= 2\left(\lambda^2 - \frac{A^2}{\rho^2}\right)\left(\frac{\partial W}{\partial I_1} + \frac{\rho^2}{\lambda^2 A^2}\frac{\partial W}{\partial I_2}\right) + W + \kappa. \end{aligned}\right\} \tag{16·1}$$

If $R_1' = R_2'$, we have
$$(W)_{\rho=r_1} = (W)_{\rho=r_2} = W_0 \text{ (say)}. \tag{16·2}$$

This result can also be obtained, for the case when $\lambda = 1$, by carrying out the integration indicated in equation (4·12) of the previous paper (Rivlin 1949*b*) and integrating the resulting equation. If, further, $R_1' = R_2' = 0$, so that the curved surfaces are force free, we have $\kappa = -W_0$.

From (15·2) and (16·1), we have

$$\Theta' = \rho\frac{\partial W}{\partial \rho} + W + \kappa. \tag{16·3}$$

Thus the resultant force F acting normally on a surface which is initially at $y = \pm b$ is given by

$$\begin{aligned} F &= 2c\lambda\int_{r_2}^{r_1} \Theta'\, d\rho = 2c\lambda\int_{r_2}^{r_1}\left(\rho\frac{\partial W}{\partial \rho} + W + \kappa\right) d\rho \\ &= 2c\lambda\Big[\rho(W+\kappa)\Big]_{r_2}^{r_1}. \end{aligned} \tag{16·4}$$

If the curved surfaces are force free so that (16·2) is satisfied and $\kappa = -W_0$, we have, from (16·4), $F = 0$; i.e. the forces acting on each of the surfaces initially at $y = \pm b$ are statically equivalent to a couple, as is evident from considerations of symmetry. The couple M acting on each of the surfaces is given, with $\kappa = -W_0$, by

$$\begin{aligned} M &= 2c\lambda\int_{r_2}^{r_1} \rho\Theta'\, d\rho = 2c\lambda\int_{r_2}^{r_1} \rho\left(\rho\frac{\partial W}{\partial \rho} + W - W_0\right) d\rho \\ &= c\lambda\left[(r_1^2 - r_2^2)\, W_0 - 2\int_{r_2}^{r_1} \rho W\, d\rho\right]. \end{aligned} \tag{16·5}$$

Since W is a function of I_1 and I_2, it is seen, from (15·2), that the relation (16·2) is satisfied if

$$A^2 = r_1 r_2/\lambda. \tag{16·6}$$

Together with (14·3), this completes the definition of the deformation (14·2), provided that the radius of one of the curved surfaces is given.

It can readily be shown, in the case when $\lambda = 1$, that the value of A^2 given by (16·6) is the only one which satisfies the relation (16·2). In this case, we have from (15·2) that

$$I_1 = I_2 = \frac{A^2}{\rho^2} + \frac{\rho^2}{A^2} + 1 = I \text{ (say)}. \tag{16·7}$$

Then, in (16·2), W is a function of I, and provided that it is a monotonic function, it follows from (16·2) that

$$(I)_{\rho=r_1} = (I)_{\rho=r_2}, \tag{16·8}$$

which in turn yields $A^2 = r_1 r_2$.

It can be shown that W is a monotonic increasing function of I, when $I_1 = I_2 = I$, by considering the simple shear of a thin sheet of the material by forces applied to the major surfaces of the sheet. For a shear of amount K, the tangential surface traction T, measured per unit area, is given (Rivlin 1948*c*, § 12) by

$$T = 2K\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right), \tag{16·9}$$

where
$$I_1 = I_2 = 3 + K^2 = I \text{ (say)}. \tag{16·10}$$

Normal surface tractions must also be applied to the major surfaces, and surface tractions must be applied to the edge surfaces. By considering the dimensions of the major surfaces to be large compared with the thickness of the sheet, the forces on the edge surfaces may be neglected in comparison with those applied to the major surfaces. In a small increase in the amount of shear from the equilibrium value K, the normal surface tractions do no work, and since the work done by the externally applied forces must be positive, T must be positive. Thus, from (16·9), we have $(\partial W/\partial I_1 + \partial W/\partial I_2) > 0$. From (16·10) we see that

$$\frac{\partial W}{\partial I} = \frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2} > 0, \tag{16·11}$$

so that, if $I_1 = I_2 = I$, W is a monotonic increasing function of I.

Returning now to the more general case when λ is not necessarily unity, we see that in the uniform extension the dimension which is initially $2b$ becomes $2b/\lambda^{\frac{1}{2}}$. If A^2 is given by (16·6), the line which is not changed in length in the flexure has in the final state of deformation a radius r_0 given by

$$r_0 = (r_1 r_2)^{\frac{1}{2}}, \tag{16·12}$$

as was shown in the previous paper (Rivlin 1949 *b*) for the case when $\lambda = 1$ and W is given by (6·11).

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